

Covariant description for superfluids in gravitational fields

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Abstract

In this paper we develop a formalism to describe a superfluid in a gravitational background. This formalism is based on a covariant generalization of the field description for a superconductor in terms of a $U(1)$ spontaneous symmetry breaking. We study the stability of the solutions for a vortexless fluid and the force acting on vortices in the fluid, which is a generalization of the well-known flat space-time Magnus force. To clarify the development we include the explicit discussion of two particular cases, one of them of astrophysical interest.

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I. INTRODUCTION

In this paper we study the behavior of a superfluid in presence of a gravitational field. Under the usual conditions that can be obtained in a laboratory only helium shows the superfluid effect, because this is the only substance that remains liquid at temperatures low enough for quantum effects to dominate. New and appealing scenarios with very interesting possibilities are provided by some astrophysical systems. One scenario is related to neutron stars, where there are very high pressures and densities, and thus where superfluid effects can appear at relatively high temperatures. A widely accepted description for the constituent matter of a neutron star considers two coexisting quantum superfluids, neutron and proton seas, where quantum dynamics plays an essential role [1]. But, in contrast with laboratories on Earth, there we have not only superfluids but also strong gravitational fields. This makes it possible to have significant gravimagnetic effects, which could produce phenomena similar to the ones that appear in the case of the superconductors [2]. We can also mention a more speculative scenario, of interest at a cosmological level, which is the formation of a condensate in the sea of relic neutrinos [3]. Scenarios like these open the possibility of gaining access to a phenomenology which would allow a deeper understanding of different aspects of the gravitational interaction in quantum systems.

There have already been some attempts to describe a covariant superfluid, usually stressing classical geometrical aspects of the problem [4]. We present here a different approach, which is a covariant generalization of the description of the superconductor phenomenology based on a $U(1)$ spontaneous symmetry breaking [5]. This approach allows us to discuss dynamical aspects of the superfluids in a gravitational background, on a very general and well established basis, with a clear phenomenological interpretation.

In the following section we develop the general formalism for a superfluid in a gravitational background, and discuss the dynamics of the scalar bosonic excitations. We make explicit the corresponding equation of motion and the conditions that warrant the stability of the ground state. In Section 3 we consider the presence of vortices and study the forces

acting on them, using a Kalb-Ramond description for the vortices [6]. In particular, we obtain the force due to the background, which is a generalization of the well known Magnus force, and the stable vortex configurations. Section 4 contains the analysis of two cases, a superfluid in a laboratory on Earth and in a neutron star, which exemplify the general formalism. Finally, in the last section we make some comments and remarks.

II. SUPERFLUID IN A GRAVITATIONAL BACKGROUND

The superfluidity phenomena is related at the microscopic level to the formation of a condensate that spontaneously breaks a global symmetry. The spontaneous breaking of the symmetry leads to a Nambu-Goldstone excitation with zero energy in the limit of vanishing momentum. We develop here a presentation for the superfluid that follows the one for superconductors in flat space-time given in Ref. [5]. We start with a fluid which is formed by particles represented by a matter field Ψ with the simplest U(1) global symmetry group $\Psi(x) \longrightarrow e^{i\Lambda} \Psi(x)$. To be specific we can think of the neutron fluid in a neutron star and the baryonic symmetry that leads to the baryon number conservation in the system. We can always perform a field splitting

$$\Psi(x) = e^{i\phi(x)} \psi(x) . \quad (1)$$

The group transformation acting on the boson ϕ is $\phi(x) \longrightarrow \phi(x) + \Lambda$.

The U(1) invariant density Lagrangian must be a function of the derivatives of ϕ and the U(1) invariant field ψ . The most general Lagrangian density allowed by the symmetries is a nonlocal function of the field, where the nonlocality extends over a range of the order of the coherence length of the superfluid. Given that we are interested in the macroscopic fluid motion, we will only consider local terms in the Lagrangian that effectively describe the long range behavior. Such terms must be scalars and should be constructed as a contraction of covariant quantities. The only possible fields involved are the gradient of ϕ , the metric and a number of tensors λ^μ , $\lambda^{\mu\nu}$, etc., which depend on the field ψ and must satisfy the additional

requirements of symmetry eventually imposed by this field. Thus, the Lagrangian can be written as the expansion

$$\mathcal{L} = \mathcal{L}_o[\psi] + \lambda^\mu[\psi]\phi_{,\mu} + \frac{1}{2}\lambda^{\mu\nu}[\psi]\phi_{,\nu}\phi_{,\mu} + \dots \quad , \quad (2)$$

where \mathcal{L}_o is a ϕ independent Lagrangian. The symmetry breaking implies that the Goldstone field ϕ has an effective dynamics below some typical energy where the condensate forms and where the tensor $\lambda^{\mu\nu}$ develops a non null expectation value. The dynamics of the superfluid is completely described by the Goldstone boson because the gap makes ψ nondynamical at low energies. However, the expectation values of the tensors $\lambda^{\mu\nu}$ and λ^μ generally depend on external forces, such as the ones induced by the gravitational field and the ones on the background state. That is, in general $\lambda^{\mu\nu}$ and λ^μ should satisfy classical equations that involve the gravitational field, of the type given by the hydrostatic equilibrium, while the Goldstone boson should satisfy the full quantum dynamics given by the Lagrangian (2). Besides this, if the Goldstone field satisfies a four dimensional dynamics the $\lambda^{\mu\nu}$ tensor must be non degenerate. The effective theory makes contact with the microscopic theory through the values of these tensor coefficients. Given a particular case these quantities could satisfy additional symmetries, which would allow ad hoc approximations to achieve a concrete physical characterization.

The ϕ field is dimensionless and thus the tensor $\lambda^{\mu_1\mu_2\cdots\mu_n}$ must have a $(4-n)$ dimension, being typically of the order of μ^{4-n} , where μ is the condensate scale. The low energy requirement states that the typical scale ϵ of $\partial_\nu\phi$ satisfies $\epsilon \ll \mu$, which justifies us in expressing the Lagrangian as an expansion on the derivatives of ϕ and in cutting this expansion at order two. In the case where the system contains several different energy scales a sensible expansion can involve different powers of the space and time derivatives [7]. This could be the case for laboratory superfluids where the Fermi momentum and the fermion masses are very different. We will focus on superfluids such as the one in a neutron star, where these quantities are similar. For the description sketched above to be valid during the whole evolution of the system, the gradient of ϕ must not only be small at a given instant, but

also at all times. From the equation of motion of ϕ , this implies that the coefficients $\lambda^{\mu\nu}$ and λ^μ must be sufficiently smooth.

The original theory is invariant under general coordinate transformations and redefinitions of the field factorization. Once the background has been established the effective theory admits only a restricted refactorization. The effective second order Lagrangian will be invariant under the field transformation $\phi \rightarrow \phi' = \phi - \Lambda$, $\psi \rightarrow \psi' = e^{i\Lambda(x)}\psi$, if $\Lambda_{,\nu}$ is of the same order as $\phi_{,\nu}$.

Inspired by the physical picture for the neutron star matter, where neutron pairs have nonvanishing expectation values, we will assume that the U(1) baryonic symmetry is broken to Z_2 , the subgroup of transformations with $\Lambda = 0$ and $\Lambda = \pi$. This is similar to the superconductivity effect, where there is a spontaneous breaking of the U(1) electromagnetic gauge symmetry to Z_2 . As ϕ parametrizes U(1)/ Z_2 , ϕ and $\phi + \pi$ are taken to be equivalent.

The baryonic current is given by

$$j^\mu = \frac{\partial L}{\partial \phi_{,\alpha}} = \lambda^{\mu\nu}(x) \phi_{,\nu} + \lambda^\mu(x) . \quad (3)$$

It contains two contributions. One is given by the Goldstone field, in terms of the derivatives of ϕ , and the other comes from the background baryonic current given by λ^μ . The continuity equation for the baryonic current, $\partial_\mu(\sqrt{g}j^\mu) = 0$, is equivalent to the equation of motion of the Nambu-Goldstone field

$$D_\mu (\lambda^{\mu\nu}(x) \phi_{,\nu} + \lambda^\mu(x)) = 0 . \quad (4)$$

The velocity of charge transport can be defined by the relation $j^\mu = n_o u^\mu$, where n_o is interpreted as the charge density in the fluid rest frame [8]. The superfluid state without vortices in a flat space-time is characterized by the relation $\nabla \times \mathbf{v} = 0$, where \mathbf{v} is the velocity of the fluid. In a curved space-time, this relation is replaced by the one obtained from taking the curl of the gradient in Eq. (3)

$$\varepsilon^{\alpha\beta\gamma\delta} D_\delta \left(\lambda^{-1} \right)_\gamma^\mu (n_o u_\mu - \lambda_\mu) = 0 . \quad (5)$$

This last equation, together with Eq. (4) defines the evolution of a vortexless superfluid in a gravitational background.

The energy-momentum tensor is

$$T_\nu^\mu = \frac{1}{2} (\lambda^{\mu\sigma} \phi_{,\sigma} \phi_{,\nu} + \lambda_\nu^\sigma \phi_{,\sigma} \phi^{,\mu}) + \frac{1}{2} (\lambda^\mu \phi_{,\nu} + \lambda_\nu \phi^{,\mu}) - g_\nu^\mu \left(\frac{1}{2} \lambda^{\sigma\tau} \phi_{,\sigma} \phi_{,\tau} + \lambda^\sigma \phi_{,\sigma} \right) . \quad (6)$$

The density of force acting on the fluid due to the interaction with the background is $f^\mu = D_\nu T^{\nu\mu} = -D_p^\mu \mathcal{L}$, where D_p^μ means the derivative with respect to the explicit dependence of the Lagrangian in x^μ . Thus

$$f^\mu = -\frac{1}{2} (\lambda^{\sigma\tau})^{;\mu} \phi_{,\sigma} \phi_{,\tau} - (\lambda^\sigma)^{;\mu} \phi_{,\sigma} . \quad (7)$$

We can define a global time by a time-like vector $\xi_\mu \sim \partial_t$, with $\xi \cdot \xi = 1$, orthogonal to space-like surfaces that are set to be equal time three-dimensional subspaces. The density of energy in such equal time space-like subspaces is $\rho_\xi = \xi_\mu T_\nu^\mu \xi^\nu$:

$$\rho_\xi = \left(\xi_\kappa \lambda^{\kappa\sigma} \xi^\tau - \frac{1}{2} \lambda^{\sigma\tau} \right) \phi_{,\sigma} \phi_{,\tau} + ((\lambda \cdot \xi) \xi^\sigma - \lambda^\sigma) \phi_{,\sigma} , \quad (8)$$

and the power exchanged with the background is given by the projection of f^μ along ξ_μ :

$$w = f^\mu \xi_\mu = -\frac{1}{2} D_\xi (\lambda^{\sigma\tau}) \phi_{,\sigma} \phi_{,\tau} - D_\xi (\lambda^\sigma) \phi_{,\sigma} , \quad (9)$$

which depends on the directional derivatives of $\lambda^{\sigma\tau}$ and λ^σ along ξ .

The covariant statement of the existence and stability of the ground state can be expressed in terms of a minimum of the energy. If we want a local extremum for $\phi_{,\sigma} = 0$, it must be

$$(\lambda \cdot \xi) \xi_\sigma = \lambda_\sigma . \quad (10)$$

This implies that ξ_σ must be proportional to λ_σ , the baryonic current for $\phi_{,\sigma} = 0$. Thus the background explicitly breaks the covariance of the theory. Furthermore, if the extremum corresponds to a minimum, the matrix

$$M^{\sigma\eta} = \frac{\partial^2 \rho_\xi}{\partial \phi_{,\sigma} \partial \phi_{,\eta}} = \left(\xi_\rho \xi^\eta g_\kappa^\sigma + \xi_\rho \xi^\sigma g_\kappa^\eta - g_\rho^\eta g_\kappa^\sigma \right) \lambda^{\rho\kappa} \quad (11)$$

must be positive definite. As stated above, if the Goldstone boson has a true four dimensional dynamics, λ_κ^ρ is regular, and thus it has one time-like eigenvector and three space-like ones. If we also require that the background be invariant under the time reversal transformation $\xi_\sigma \rightarrow -\xi_\sigma$, the $M^{\sigma\eta}$ matrix becomes reducible into two non null submatrices: a one-dimensional one in the subspace spanned by ξ_σ

$$\xi_\sigma M^{\sigma\eta} \xi_\eta = \xi_\sigma \lambda^{\sigma\eta} \xi_\eta, \quad (12)$$

and a three-dimensional one in the space-like subspace orthogonal to ξ_σ

$$(\xi^\rho \xi_\sigma - g_\sigma^\rho) M^{\sigma\eta} (\xi^\tau \xi_\eta - g_\eta^\tau) = -(\xi^\rho \xi_\sigma - g_\sigma^\rho) \lambda^{\sigma\eta} (\xi^\tau \xi_\eta - g_\eta^\tau). \quad (13)$$

Therefore, in this case to have an energy minimum, the component $\xi_\sigma \lambda^{\sigma\eta} \xi_\eta$ of $\lambda^{\sigma\eta}$ must be positive, whereas the matrix of the components in the orthogonal subspace must be negative definite. Thus the minimum energy condition guarantees dispersion relations with real propagation velocities for the low-energy excitations. If the space-like submatrix is completely degenerate, the propagation velocities are the same in all the spatial directions and we have an isotropic background. Otherwise we have an anisotropic one, with different propagation properties for the low energy excitations according to the different spatial eigenvectors of $\lambda^{\sigma\eta}$.

The spacial components u_i of the velocity are proportional to $\frac{1}{n_o} \lambda_i^\mu \phi_{,\mu}$ and this implies that the low energy expansion is also a small velocity expansion $u_i \ll 1$.

We can implement a hydrodynamical description of the fluid by writing the energy momentum tensor in terms of the current. It reads

$$T_\nu^\mu = \frac{n_o}{2} \left(u^\mu (\lambda^{-1})_{\rho\nu} (n_o u^\rho - \lambda^\rho) + u_\nu (\lambda^{-1})_\rho^\mu (n_o u^\rho - \lambda^\rho) \right) - g_\nu^\mu \frac{1}{2} (\lambda^{-1})_{\rho\tau} (n_o^2 u^\tau u^\rho - \lambda^\tau \lambda^\rho). \quad (14)$$

This tensor only contains the terms corresponding to the dynamics of the excitations with respect to the background. To have the energy momentum tensor of the fluid as a whole, it is necessary to add the contribution of the rest of the fluid. Assuming that the fluid is an isotropic perfect one up to first order in the velocities, we have

$$\mu = \frac{(\rho + p)}{n_o} = \frac{n_o}{2} \left(\lambda^{-1} \right)_i^i, \quad (15)$$

where μ is (for zero temperature) the chemical potential of the global conserved number, ρ is the energy density and p is the pressure of the fluid.

III. VORTICES AND THE GENERALIZED MAGNUS FORCE

We have not considered the existence of vortices up to now, but they will be present if the fluid has some angular momentum. In general, the superfluid cannot rotate as a whole, and therefore if it has a non null angular momentum it must be supported by vortices. It is difficult to analyze the vortex-fluid interaction in the Goldstone field representation for the superfluid, because the vortex imposes non trivial boundary conditions to the scalar field. In the large distance limit where the vortex size could be considered negligible these conditions manifest themselves as a field nondifferentiability that leads to a noncommutativity of its partial derivatives. A more suitable approach could be developed relying on the dual description of the scalar field in terms of a Kalb-Ramond one, an antisymmetric tensor of second order $B_{\mu\nu}$ [6,9]. Both massless theories, a k-form field and a (n-k-2)-form field, where n is the space-time dimension, are equivalent not only at a classical level, but also at the quantum one. This point will be discussed in detail elsewhere [10]. We will use a generalized version of this duality here.

The dual variables are defined by expressing the Noether conserved baryonic current as a topologically conserved current. This is achieved with the identification

$$J^\lambda = \frac{1}{6} E^{\mu\nu\rho\lambda} H_{\mu\nu\rho}, \quad (16)$$

where $H_{\mu\nu\sigma} = \partial_\mu B_{\nu\sigma} + \partial_\nu B_{\sigma\mu} + \partial_\sigma B_{\mu\nu}$ and E is the Hodge tensor with covariant and contravariant components $E_{\mu\nu\rho\lambda} = \sqrt{-g} \varepsilon_{\mu\nu\rho\lambda}$ and $E^{\mu\nu\rho\lambda} = -\frac{1}{\sqrt{-g}} \varepsilon^{\mu\nu\rho\lambda}$ respectively. The Levi-Civita symbols denote the index permutation signature. The completely antisymmetric tensor E commutes with the covariant derivatives. The inverse relation to (16) is $H_{\mu\nu\rho} = -E_{\mu\nu\rho\lambda} J^\lambda$.

The divergence of the current J^λ is proportional to the exterior derivative $d \wedge H$, and nullifies because $H = d \wedge B$. In components it reads

$$\varepsilon^{\mu\nu\rho\lambda} \partial_\lambda H_{\mu\nu\rho} = 0 . \quad (17)$$

In this way the equation of motion for the field ϕ , which is equivalent to the baryonic current conservation, is satisfied automatically in a topological sense in the Kalb-Ramond field scheme.

In a similar way, the topological property of the scalar field $d \wedge (d \wedge \phi) = 0$ becomes the equations of motion for the Kalb-Ramond field $B_{\mu\nu}$, by means of the duality transformations. Duality states a correspondence between the physical solutions of the two theories. This correspondence is realized as a canonical transformation at the level of Dirac brackets, which define the quantum theory [10].

Both descriptions are linked by Eq. (16), which is a relation between the field derivatives

$$\phi_{,\kappa} = \left(\frac{1}{6} E^{\mu\nu\rho\sigma} H_{\mu\nu\rho} - \lambda^\sigma \right) \lambda_{\sigma\kappa}^{-1} , \quad (18)$$

where the tensor $\lambda_{\sigma\kappa}^{-1}$ is the inverse of $\lambda^{\sigma\kappa}$. This relation is a generalization of the one used in Ref. [6], where only the particular case $\lambda_{\sigma\kappa} \propto g_{\sigma\kappa}$ is considered. The curl of $\phi_{,\kappa}$ is null, and this implies that the curl of the right hand side must also be null. This gives the equations of motion for the B field:

$$E^{\alpha\beta\gamma\kappa} D_\gamma \left(\frac{1}{6} E^{\mu\nu\rho\sigma} H_{\mu\nu\rho} \lambda_{\sigma\kappa}^{-1} - \lambda_{\sigma\kappa}^{-1} \lambda^\sigma \right) = 0 . \quad (19)$$

The corresponding action that leads to this equation can be written as

$$\mathcal{S}_B = \frac{\kappa}{3} \int d^4x \sqrt{-g} H_{\alpha\beta\gamma} E^{\alpha\beta\gamma\kappa} \left(\frac{1}{12} E^{\mu\nu\rho\sigma} H_{\mu\nu\rho} \lambda_{\sigma\kappa}^{-1} - \lambda_{\sigma\kappa}^{-1} \lambda^\sigma \right) . \quad (20)$$

Equating the energy-momentum tensor from this action with the one for the scalar field, Eq. (6), using the relation (18), we get $\kappa = -\frac{1}{2}$.

The 2-form Kalb-Ramond field naturally couples with the two dimensional manifold generated by the world sheet of the vortex moving along time. This formalism is advantageous because it allows us to describe the fluid-vortex interaction with an explicit term in

the action, without using any extra boundary conditions. Using the antisymmetric tensor $J^{\mu\nu} = \eta \int d\sigma^{\mu\nu} \delta^4(x - y(\sigma, \tau))$, which describes the surface that sweeps the vortex in space time, with the coordinates (σ, τ) parametrizing the world sheet, the fluid-vortex interaction term is

$$\mathcal{L} = B_{\mu\nu} J^{\mu\nu} , \quad (21)$$

a metric independent term because it corresponds to the surface integral of a two form. It is possible to think of different interaction terms that depend on the metric or that have field derivatives, but for dimensional reasons those terms should be suppressed. They must contain additional powers of the superfluid coherence length over the vortex curvature scale or the space curvature, and can be neglected in this context.

Taking into account the interaction with the vortex, the equation of motion is written

$$\frac{1}{2} E^{\alpha\beta\gamma\kappa} D_\gamma \left(\frac{1}{6} E^{\mu\nu\rho\sigma} H_{\mu\nu\rho} \lambda_{\sigma\kappa}^{-1} - \lambda_{\sigma\kappa}^{-1} \lambda^\sigma \right) = J^{\alpha\beta} . \quad (22)$$

This equation describes the possible singularities of the Goldstone field since the first term translates as the curl of the gradient of ϕ . Using the Stokes theorem, the flux of $J^{\alpha\beta}$ is related with the circulation $\oint \nabla\phi ds = k\pi$ around the vortex lines, where k is an integer. This condition is imposed in the Kalb-Ramond field scheme by $\eta = k\frac{\pi}{4}$, where k is the vortex topological number.

Equation (22) provides the general relation between current and vorticity

$$\frac{1}{2} E^{\alpha\beta\gamma\kappa} D_\gamma \left(\lambda_{\sigma\kappa}^{-1} (J^\sigma - \lambda^\sigma) \right) = J^{\alpha\beta} , \quad (23)$$

and allows us to infer the vortex distribution from the current. Using the Stokes theorem the circulation $\frac{1}{2} \oint dx^\kappa \lambda_{\sigma\kappa}^{-1} (J^\sigma - \lambda^\sigma)$ in the perimeter is equal to the flux $\int dx^\mu dx^\nu E_{\mu\nu\alpha\beta} J^{\alpha\beta}$ over the enclosed surface, and we can write

$$\oint \lambda_{\sigma\kappa}^{-1} (J^\sigma - \lambda^\sigma) dx^\kappa = \pi N_v , \quad (24)$$

where the second term is the flux of $J^{\alpha\beta}$, and represents the total number of vortices enclosed in the integration path.

The presence of a vorticity gives an additional term to the force acting on the fluid. The force is given by

$$\tilde{f}^\nu = D_\mu T^{\mu\nu} = -J_{\mu\sigma} H^{\mu\sigma\nu} - J^\kappa D^\nu \left(\lambda_{\sigma\kappa}^{-1} \lambda^\sigma \right) + \frac{1}{2} J^\kappa J^\sigma D^\nu \left(\lambda_{\sigma\kappa}^{-1} \right) , \quad (25)$$

where the first term on the right hand side comes from the new term in the equation of motion for H , and the Bianchi identities have been used. The last terms are only relevant in regions where the background has a large gradient, with a scale of the order of the coherence length, as happens for the dynamics of the vortices in the crust of a neutron star where the vortex pins.

We will now calculate the force acting on a single vortex. In this case the solution of (22) could be written as a background solution, whose source are the other vortices, $B_o^{\mu\nu}$, plus a particular solution with the vortex as a source, $B_v^{\mu\nu}$, i.e. $H^{\mu\sigma\nu} = H_o^{\mu\sigma\nu} + H_v^{\mu\sigma\nu}$. The term $H_o^{\mu\sigma\nu}$ can be written using relation (16), from the average current

$$H_o^{\mu\sigma\nu} = -E^{\mu\sigma\nu\rho} J_\rho . \quad (26)$$

We are not taking into account the dynamics of the vortex, which could be responsible for tension effects but is not given by the effective theory. These effects are related to interaction terms between $J_{\mu\sigma}$ and $H_v^{\mu\sigma\nu}$, which are not relevant if the curvature radius of the vortex is much greater than the coherence length. Considering also that the background potentials have a slow variation along the vortex size, the force on the vortex per unit length becomes

$$f^\nu = -\tilde{f}^\nu \simeq -J_{\mu\sigma} E^{\mu\sigma\nu\rho} J_\rho . \quad (27)$$

In general for a vortex in equilibrium the Magnus force compensates the total external forces acting on it. Considering only the interactions here discussed, the equilibrium is specified by the null Magnus force condition, which can be put in terms of the current as

$$J^\kappa \left[D_\delta \left(\lambda_{\sigma\kappa}^{-1} (J^\sigma - \lambda^\sigma) \right) - D_\kappa \left(\lambda_{\sigma\delta}^{-1} (J^\sigma - \lambda^\sigma) \right) \right] = 0 . \quad (28)$$

The vortex can be locally characterized by its tangent vector m^ν and its quadrivelocity v^ν . Thus we have

$$d\sigma^{\mu\nu} = (v^\mu m^\nu - m^\mu v^\nu) d\sigma d\tau , \quad (29)$$

where σ is defined such that $m^\nu m_\nu = 1$ and τ is the proper time on the vortex, so $v^\nu v_\nu = -1$. Using $J^\mu = n_0 u^\mu$, where u^μ is the fluid quadrivelocity, and integrating over a transversal cut, the force per unit length finally results

$$f^\mu = -\frac{\pi}{2} n_0 \sqrt{-g} g^{\mu\rho} \epsilon_{\rho\nu\sigma\tau} u^\nu v^\sigma m^\tau . \quad (30)$$

This is the expression for the generalized Magnus force. Introducing $\delta u^\mu = v^\mu - u^\mu$ and considering that to first order in the velocities $\delta u^0 = 0$, the 3-d force acting on a vortex with $m^\mu = (0, \mathbf{m})$ is

$$\mathbf{f} = \frac{\pi}{2} n_0 \sqrt{-g} \sqrt{-g^{00}} [\mathbf{G} \cdot (\delta \mathbf{u} \times \mathbf{m})] , \quad (31)$$

where $\mathbf{G} = (g^{ij})$, and $\mathbf{f} = (f^i)$, and the power dissipated by the interaction of the vortex with the fluid is

$$f^0 = \frac{\pi}{2} n_0 \sqrt{-g} \sqrt{-g^{00}} \left(\sqrt{-g^{00}} \mathbf{v} \cdot \mathbf{u} \times \mathbf{m} + 2 \mathbf{g} \cdot \delta \mathbf{u} \times \mathbf{m} \right) . \quad (32)$$

In flat space-time this force reduces to the ordinary Magnus force $\mathbf{f} = \frac{\pi}{2} n_0 (\delta \mathbf{u} \times \mathbf{m})$.

The Magnus force implies that the equilibrium is reached when the vortex moves at the same velocity as the medium, or when the vortex is aligned with the relative velocity between the vortex and the background.

IV. EXAMPLES

In this section we present two examples of superfluids in a gravitational field. The first is a superfluid in a laboratory on Earth. It is simple, but its relevance is only academic because the interesting effects are extremely weak. On the other hand, the second example is of great phenomenological interest, because it corresponds to a superfluid in a neutron star, which is one of its main components.

In the construction of the effective Lagrangian theory we include all the terms consistent with the symmetries of the problem and relevant for the energy scale that matters. In

general there is more than one Goldstone boson that arise from the symmetry breaking. In the following examples we consider only one boson. This is true for He^4 superfluid but not for He^3 or the neutron star case, where part of the superfluid can be in the spin two 3P_2 state. In spite of this the baryonic phase can be factorized from the spin degrees of freedom. Specifically, for the neutron star the effect of the interaction of the baryonic Goldstone boson with these spin degrees is negligible [1]. This is easy to see by considering that the ratio between the spin angular momentum and the orbital angular momentum for a single neutron is $\frac{\hbar}{\Omega R^2 m} \ll 1$, indicating that the spin can not significantly alter the global dynamics of the baryonic current.

A. Superfluid in a laboratory on Earth

In this case we have a weak gravitational field without a noticeable contribution from the fluid itself, with a time independent metric that can be written:

$$g^{00} = 1 - 2U , \quad (33)$$

$$g^{ij} = - (1 + 2U) \delta^{ij} , \quad (34)$$

$$g^{0i} = h^i , \quad (35)$$

where $U = g z$ is the Newtonian potential and \vec{h} gets contributions from the dragging and inertial effects due to the Earth rotation. The existence of an energy extremum for a null ϕ gradient implies that λ_μ is proportional to the time derivative Killing vector, with $\lambda_0 = n(z)$ and $\lambda_i = 0$. Here the minimum energy condition is equivalent to $\lambda^{00} > 0$ and to the matrix λ^{ij} negative definite. In general both the gravitational dragging and the fluid velocities are small, so that the second order tensor $\lambda^{\mu\nu}$ can be considered independent of \vec{h} . Taking into account the rotational symmetry on the vertical \hat{z} axis, the traslational symmetry in the horizontal (x, y) plane, and the invariance under temporal inversion, the non null components are

$$\lambda_0^0 = a(z) , \quad \lambda_x^x = \lambda_y^y = \lambda_z^z = b(z) . \quad (36)$$

Here a and b are scalars that depend only on the fluid conditions. We take $\lambda_x^x = \lambda_y^y = \lambda_z^z$ because the gravitational field scale is much longer than the microphysics scale that determines the superfluid state, so that any possible microscopic anisotropy induced by the gravitational field should be negligible. The current can be written as

$$j_0 = a\dot{\phi} + n \quad , \quad j_i = b\partial_i\phi . \quad (37)$$

Let us consider the stationary case. The charge density in the fluid rest frame n_o is equal to n at first order in the velocity. The tridimensional velocity is given by the contravariant components

$$\vec{v} = \frac{1}{n_o}(j^i) = \vec{h} - \frac{b}{n_o}(1 + 2U)\vec{\nabla}\phi . \quad (38)$$

This implies $\vec{\nabla} \times \left(\frac{n_o(1-2U)}{b}(\vec{v} - \vec{h}) \right) = 0$, a generalization of the usual superfluid relation $\vec{\nabla} \times \vec{v} = 0$. This equation is analogous to the superconductor equation $\vec{\nabla} \times (\vec{v} - \frac{e}{m}\vec{A}) = 0$, but there is a subtle detail in this analogy. In our case the origin of the velocity drift is the metric and not the connection, as in the superconductor case. Thus for a superfluid the velocity field in the fundamental state will not be null, but equal to \vec{h} . The term with the gradient of the field in Eq. (38) is non null when there are vortices contributing to the fluid rotation. As can be deduced from this equation the contribution of \vec{h} generally diminishes the number of vortices with respect to a flat space situation.

When \vec{h} has an inertial origin, due to choosing a rotating reference frame, we have $\vec{\nabla} \times \vec{h} = -\vec{\omega}$, the angular velocity vector. This tells us that the superfluid is in fact non rotating. Inertial effects on superfluids have been used in the construction of a precision gyrometer [11]. In the case of a superconductor, inertial effects lead to a compensating London field [12], which is also induced by the gravitational dragging [13].

The effective mass of the boson condensate m^* can be identified from Eq. (38) at null gravitational field, as the coefficient of $\vec{\nabla}\phi$. It results $m^* = n_o/b$. The Lagrangian density is

$$L = (1 - 2U) \left(a\dot{\phi}^2 - b(\vec{\nabla}\phi)^2 + n\dot{\phi} \right) + (2a\dot{\phi} + n) (\vec{h} \cdot \vec{\nabla}\phi) - 4bU (\vec{\nabla}\phi)^2 . \quad (39)$$

The limit of zero gravitational field allows us to clarify the physical meaning of n , a and b , by comparing with the low energy BCS Lagrangian [14]. Thus $n = k_F^3/3\pi^2$, with k_F the Fermi wave number, is the charge density. We can also relate a and b with the Fermi wave number and the fermion mass m , $a = mk_F/2\pi^2$ and $b = k_F^3/6\pi^2m$, and thus the sound velocity is given by $\sqrt{\frac{b}{a}} = \frac{k_F}{m\sqrt{3}} = v_F/\sqrt{3}$, where v_F is the Fermi velocity. The relation between the fermion mass and the effective pair mass is $m^* = 2m$. Finally, the factor $(1 - 2U)$ corresponds to a redshift effect.

If there are vortices, the expression (30) gives the Magnus force acting upon them. In this case we have the usual flat space-time force only affected by a redshift factor.

B. Superfluid neutron star

Here we develop with some detail the example of the superfluid component in a neutron star. In the case of a neutron star at rest we can assume that the metric satisfies a spherical symmetry

$$(ds_o)^2 = e^{2\Phi} dt^2 - e^{2\Lambda} dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2) . \quad (40)$$

The potentials $\Phi(r)$ and $\Lambda(r)$ that characterize this metric are given by the Oppenheimer-Volkoff equations for cold stars [15].

In the case of neutron stars the friction is high enough to drive the star to equilibrium in relatively short times. This allows us to assume a stationary neutron star, which rotates at a small angular velocity Ω . This magnitude represents the angular velocity measured by an observer at rest with the fluid, and is related with the fluid velocity by $\Omega = u^\varphi/u^t$, such that

$$j^\mu = n_o u^0(1, 0, 0, \Omega) . \quad (41)$$

We can make an expansion of the rotating star metric in the perturbation with respect to the rest star metric and the angular velocity Ω . The source for this perturbation is the

superfluid energy momentum tensor. It can be considered a small quantity for ordinary pulsars because the gravitational acceleration, approximately $\frac{GM}{R^2}$, is much greater than the centrifugal one, $\Omega^2 R$. As an example, for the rapidly rotating pulsar Crab the quotient of these accelerations is approximately 50, implying small deformations. Hence we can keep the expansion up to the first order in the fluid angular velocity [16]

$$ds^2 = (ds_o)^2 - 2r^2 \sin^2 \theta \omega d\varphi dt . \quad (42)$$

The gradients $\phi_{,\nu}$ can also be considered first order quantities because they can be chosen to be zero for the star at rest.

In the construction of the effective Lagrangian theory we will consider all the terms that satisfy the symmetries of the problem and are relevant for the energy scale that matters. We can identify several symmetries: the baryonic global symmetry, the covariance under general coordinate transformations, the spherical symmetry, the temporal translation and the temporal reflection symmetry of the star at rest.

The time translation and rotational invariance makes $\lambda_\mu = f(r) \chi_\mu$, where $\chi_\mu = (e^{2\Phi}, 0, 0, 0)$ is the time Killing vector. The covariant components of this Killing vector are non dynamical, contrary to the contravariant components, which depend on the metric perturbations.

Time translation and spherical symmetry impose important restrictions to $\lambda^{\mu\sigma}$. We must have

$$\lambda_0^0 = a(r) , \quad \lambda_r^r = b(r) , \quad \lambda_\theta^\theta = c(r) , \quad \lambda_\varphi^\varphi = c(r) , \quad (43)$$

where a , b and c are scalar quantities, because they are the eigenvalues of a mixed tensor. The stability of the ground state requires that a , b , and c must be positive. The metric perturbations do not affect the quadratic term in $\phi_{,\sigma}$ because this is already of second order.

The interpretation of b and c comes from the equations

$$u_r = \frac{b(r)}{n_0(r)} \partial_r \phi = \frac{1}{m_r^*(r)} \partial_r \phi , \quad (44)$$

$$u_\theta = \frac{c(r)}{n_0(r)} \partial_\theta \phi = \frac{1}{m_\theta^*(r)} \partial_\theta \phi , \quad (45)$$

$$u_\varphi = \frac{c(r)}{n_0(r)} \partial_\varphi \phi = \frac{1}{m_\varphi^*(r)} \partial_\varphi \phi , \quad (46)$$

where $m_r^* = n_0/b$, $m_\theta^* = m_\varphi^* = n_0/c$ are scalar quantities of dimension one that depend only on the fluid conditions, and because of this are functions of the radius. By using the equivalence principle, these parameters can be interpreted as the effective mass of the bosonic quasiparticles in the different directions. This is the double of the fermionic particle effective mass, which can be calculated from the particular model as the ratio between the Fermi momentum and the Fermi velocity.

The quantities that determine the fluid conditions are related to each other independently of the gravitational field. This is because this field can be considered constant along the microscopic nuclear interaction range that determines the plasma dynamics. Because of this, and as long as the Fermi surface of the fluid has no preferred direction, we can take $m_r^* = m_\theta^* = m_\varphi^* = m^*$. For a neutron star m^* includes plasma effects and has values of the order of twice the neutron mass [17].

For the star at rest we can identify $j_0 = \lambda_0 = n_o e^\Phi$. Up to first order in ω , for a rotating star u_0 is the same as for a star at rest, and thus this equation is also valid for a rotating star up to first order in the angular velocity.

The relation between the angular velocity Ω and the metric coefficient ω in absence of vortices is given by Eq.(5). The solution of this equation is $\Omega = \omega(r, \theta)$, i.e. the angular velocity of the fluid must be the same as the angular velocity that takes an object that falls free from infinity to the point (r, θ) , and corresponds to the angular velocity of local inertial frames with respect to the fixed stars.

Furthermore, in the case we are discussing here, the star that produces the gravitational field is formed by the rotating fluid. Thus ω must satisfy the Einstein equation for the $R_{\varphi t}$ Ricci tensor component, with the energy momentum component $T_{\varphi t}$ of the fluid as the source. The solution for the perturbed Einstein equation is a θ -independent ω that satisfies

$$\begin{aligned}
& \omega_{,rr} + \left(\frac{4}{r} - \Lambda' - \Phi'\right) \omega_{,r} + \frac{2}{r} \left(\frac{1}{r} + \Phi' - \Lambda' - \frac{1}{r} e^{2\Lambda}\right) \omega \\
& = 8\pi e^{2\Lambda} ((\rho + 3p) \omega - 2(\rho + p) \Omega) .
\end{aligned} \tag{47}$$

For nonsingular gravitational potentials Λ and Φ the only solution for $\Omega = \omega$ with regular geometry is $\omega = 0$ [2]. That means that there is no rotating star solutions without vortices, i.e. a rotating superfluid star necessarily contains vortices, in which case $(\Omega - \omega)$ is non null. The background acts on these vortices with a force given by Eq. (27). If this is the only force upon the vortex, from it we can obtain the vortex profile in the stationary state. In this case the vortex adapts to the background in such a way that this force becomes null. This configuration is given by Eq. (28), and in the case we are discussing here it is consistent with $j^0 = \lambda^0 = n_o e^{-\Phi}$ and $j^\varphi - \lambda^\varphi = n_o e^{-\Phi} (\Omega - \omega)$. Furthermore, Eq. (23) gives the relation between the current j^μ and the vorticity $J^{\mu\nu}$, and allows us to compute the distribution and orientation of the vortices. According to Eq. (24), in this case we have

$$\Omega - \omega(r, \theta) = \frac{N_v(r, \theta)}{m^* e^{-\Phi(r)}} \frac{1}{r^2 \sin^2 \theta} , \tag{48}$$

where N_v is the number of vortices within a closed circular path that passes by the point (r, θ) and is perpendicular to the \hat{z} axis, m^* is the effective mass, and $e^{-\Phi}$ gives the red shift of the effective mass due to the gravitational field.

For a rotating star at the minimum energy configuration Ω is constant, and thus Eq. (48) together with Eq. (47) allows us to completely determine the vortex profile. The angle β between the axis of the star and the direction of the vortex lines at a given point (r, θ) can be deduced from Eq. (48), using the cylindrical symmetry of the system. It reads

$$\sin \beta = \frac{1}{2} \frac{\kappa(r) \sin(2\theta)}{(1 + \kappa(r)(\kappa(r) - 2) \sin^2(\theta))^{1/2}} , \tag{49}$$

where $\kappa(r) = \frac{r}{2} \left(\frac{1}{\Omega - \omega} \frac{d\omega}{dr} - \frac{1}{m^*} \frac{dm^*}{dr} + \frac{d\Phi}{dr} \right)$. The first term in κ corresponds to a purely geometrical contribution due to the Lense-Thirring effect, whereas the two last terms depend on the microscopic structure of the fluid and the star as a whole.

In particular, if we take r equal to the radius R of the star and $\theta = \frac{\pi}{2}$, we have $\omega = \frac{2GJ}{R^3}$,

where J is the angular momentum of the star, and thus the total number of vortices contained by the star is

$$N = m^* e^{-\Phi(R)} \left(\Omega R^2 - \frac{2GJ}{R} \right). \quad (50)$$

A noticeable effect of the gravitational background is a decrease of the vortex density with respect to a similar situation in flat space-time. In the case of a typical neutron star this decrease is of the order of 15%.

V. FINAL REMARKS

We have considered the superfluidity phenomena in presence of a gravitational background. Our starting point is the spontaneous symmetry breaking of a U(1) symmetry. This field-theoretical approach allows us to develop from first principles a fully covariant formalism. Within this framework we study general aspects of the dynamics of the superfluid in a gravitational field. We analyze the force acting on a vortex, which is a generalization of the well known Magnus force, and in particular we find the profile of a vortex in equilibrium with the condensate. This approach makes contact with the microscopic theory which describes the details of the superfluidity phenomena in two points. One is a very basic and general one, with a strong theoretical support, which is the spontaneous symmetry breaking mechanism. The other point of contact is the specific description of the background through the tensors $\lambda^{\mu\nu}$ and λ^μ , which in general must satisfy some symmetry requirements, but their details depend on the microscopic physics of the system. In some relatively simple cases it is possible to construct a rather closed description, dependent only on a few phenomenological parameters, making use of the known symmetries.

Our results are exemplified with two systems. One of them is a superfluid in a terrestrial laboratory, where we have a weak gravitational field. The main result here is the formal analogy with a superconductor, with the gravimagnetic field \vec{h} playing the role of the vector potential \vec{A} , and the Newtonian potential U the role of the Coulomb potential. This analogy

is not complete, because the geometrical origin of these fields is different. In the gravitational case \vec{h} is introduced by the metric, whereas the \vec{A} field is due to the connection. One consequence of this difference is that in the electromagnetic case we have a Meissner effect, while in the gravitational case we have an anti-Meissner one [2].

The other example considers a more interesting system, which is the superfluid in a neutron star. In this case we have a strong gravitational field, which makes a fully covariant treatment unavoidable. Here we construct a description of the system based on the symmetries of the star, and this is enough to determine the shape and distribution of the vortices, assuming that there are no external forces. This should be the case in the star core where there are no pinning forces. Other forces, such as the magnetic ones, cannot appreciably alter the vortex distribution because their energy is much smaller than the rotational energy. The generalized Magnus force here analyzed could be very relevant for a detailed study of the transient during the pulsar glitches, if we have an adequate model for the pinning forces.

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